FRACTION-LIKE RATES FOR PREFERENTIAL VOTING

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Abstract

A method is given for determining a mixed social choice out of a paired-comparison matrix. The method combines a projection procedure introduced in previous papers of the same authors and a classical method due to Zermelo. The resulting method is proved to have certain desirable properties, which include: compliance with a majority principle, clone consistency, and continuity of the mixing fractions with respect to the data.

Keywords: *preferential voting, paired comparisons, Condorcet, majority principle, clone consistency, continuous rating, one-dimensional scaling, Zermelo's method of strenghts, Luce's choice model.*

AMS subject classifications: *05C20, 91B12, 91B14, 91C15, 91C20.*

A vote is an expression of the preferences of several individuals about certain options with a view towards reaching a common decision. Generally speaking, the decision need not be choosing a single option, but it can also take the form of mixing several options according to certain proportions. For instance, parliamentary elections based on closed party lists can be seen from that point of view. This article is aimed at methods for suitably determining the fractions of such mixed social choices.

The input from which we are to derive these fractions is the matrix of preference scores of Ramon Llull and Condorcet [9: §3, §7], i.e. the matrix that compares each option to every other in terms of the number of voters who prefer the former to the latter. In the following, we will refer to such a matrix as a Llull matrix. Our problem is a special case of a more general one where a paired-comparison matrix is to be summed up into a set of priority rates (not necessarily with the character of mixing fractions). Such problems arise not only in voting theory, but also in multi-criteria decision theory, psychometrics, sport tournaments, web search engine rankings, etcetera.

Besides being especially interested in fraction-like rates, we are also committed to comply with certain majority principles that are especially relevant in the case of voting. On the other hand, we are also especially interested in being able to include the case of incomplete preferences, that is, the case where the preference scores for an ordered pair of options and its opposite are not fully complementary to each other. This is of interest since the practical implementation of preferential voting often allows the individual votes to be silent about some pairs of options. Being able to deal with such situations involves certain difficulties which are not present in the complete case.

Our results have to do with the compatibility between two particular rating methods. One of them is essentially an extension, to the incomplete case, of the rank-based count of Nikolaus von Kues and Jean-Charles de Borda [9: § 4, § 5]. The other is a celebrated method introduced in 1929 by Ernst Zermelo [13]. In the following we will refer to them respectively as the method of **mean preference scores** and the method of **strengths**. In a general incomplete situation they can easily disagree with each other in the ordering of the options. However, such incompatibilities disappear when the Llull matrix has a certain special structure that we will refer to as **CLC structure**. This structure will hardly be present in the original Llull matrix, but it is always present after one applies the **CLC projection** procedure introduced in [1, 2].

As it was shown in those articles, the CLC projection has the virtue that a subsequent application of the method of mean preference scores results in a rating method that combines several good properties. However, the resulting rates are not suitable for defining the fractions of a mixed social choice. In fact, they fail at certain conditions that seem very appropriate for that purpose. More specifically, they fail at the following one: If a subset of options has the property that every one of them is unanimously preferred to any other from outside that subset, then the mixing fractions must vanish outside of that subset. As we will see, a condition of this kind is a characteristic property of the method of strengths.

By putting together these facts, namely, the good properties of the CLC projection followed by the method of mean preference scores, the compatibility between mean preference scores and strengths when the matrix has a CLC structure, and the suitability of the strengths as mixing fractions, we will obtain a rating method that combines the following properties: fraction character, compliance with a majority principle, clone consistency, and continuity with respect to the data. To our knowledge, the existing literature does not offer any other rating method with these properties.

1   Llull matrices and priority ratings

1.1   We consider a finite set A. Its elements represent the options which are the matter of a vote. The number of elements of A is N.

By a matrix of **preference scores**, or **Llull matrix** we mean a mapping v whereby every ordered pair of different options x and y is assigned a number v_{xy} in the interval $[0, 1]$, these numbers being restricted to satisfy

$$v_{xy} + v_{yx} \leq 1. \tag{1}$$

The number v_{xy} measures how much is x preferred to y; in particular, $v_{xy} = 1$ means that x is fully preferred to y, whereas $v_{xy} = 0$ means a total lack of preference for x over y. The restriction (1) obeys to the asymmetric character of the notion of preference; in particular, x being fully preferred to y is incompatible with any amount of preference for y over x. Having $v_{xy} + v_{yx} = 1$ means a complete information about the preference between x and y, whereas $v_{xy} + v_{yx} = 0$ means a total lack of information about it. A matrix of preference scores satisfying $v_{xy} + v_{yx} = 1$ for any x and y will be said to be **complete**.

Besides the scores v_{xy}, in the sequel we will often deal with the **margins** m_{xy} and the **turnouts** t_{xy}, which are defined respectively by

$$m_{xy} \; = \; v_{xy} - v_{yx}, \qquad t_{xy} \; = \; v_{xy} + v_{yx}. \tag{2}$$

Obviously, their dependence on the pair xy is respectively antisymmetric and symmetric, that is

$$m_{yx} \; = \; -m_{xy}, \qquad t_{yx} \; = \; t_{xy}. \tag{3}$$

It is clear also that the scores v_{xy} and v_{yx} can be recovered from m_{xy} and t_{xy} by means of the formulas

$$v_{xy} \; = \; (t_{xy} + m_{xy})/2, \qquad v_{yx} \; = \; (t_{xy} - m_{xy})/2. \tag{4}$$

Instead of the margins $m_{xy} = v_{xy} - v_{yx}$, sometimes, especially in decision theory, one considers the **ratios** $p_{xy} = v_{xy}/v_{yx}$ (which requires the preference scores to be all of them positive). Alternatively, one can consider the **relative scores** $q_{xy} = v_{xy}/t_{xy}$ (which only requires the turnouts to be positive). Obviously, the matrix of relative preference scores is always complete. The ratios and the relative scores are related to each other by the formulas $p_{xy} = q_{xy}/(1 - q_{xy})$, $q_{xy} = p_{xy}/(1 + p_{xy})$. Notice however that in the incomplete case neither the margins, nor the ratios, nor the relative scores, allow to recover the original scores, unless one knows also the turnouts t_{xy}.

In order to refer to it as a whole, the Llull matrix made of the preference scores v_{xy} will be denoted as (v_{xy}), or alternatively as V. We will also use the notation V_{RS} to mean the restriction of (v_{xy}) to $x \in R$ and $y \in S$, where R and S are arbitrary non-empty subsets of A. Similarly, if (u_x) is a collection of numbers indexed by $x \in A$, its restriction to $x \in R$ will be denoted as u_R.

1.2 The simplest and most natural rate of the overall acceptance of an option x is its **mean preference score**, that is, the arithmetic mean of its preference scores against all the other options:

$$\rho_x \;=\; \frac{1}{N-1} \sum_{y \neq x} v_{xy}. \tag{5}$$

This quantity is linearly related to the rank-based count proposed in 1433 by Nikolaus von Kues [9: §1.4.3, §4] and again in 1770–1784 by Jean-Charles de Borda [9: §1.5.2, §5] (both of them being restricted to the complete case). More specifically, their count amounts to $1 + (N-1)\rho_x = (1 - \rho_x) + \rho_x N$. Instead of it, in [1, 2] we considered the **mean ranks** $\bar{r}_x$, which are given by

$$\bar{r}_x \;=\; N - (N-1)\,\rho_x \;=\; \rho_x + (1 - \rho_x)\,N. \tag{6}$$

Notice that, contrarily to ρ_x, lower mean ranks mean a higher acceptance. The **rank-like rates** R_x proposed in [1, 2] are nothing else than the mean ranks that are obtained after transforming the Llull matrix according to the CLC projection procedure.

1.3 In this paper we are especially interested in fraction-like rates, i. e. rates suitable to define the fractions of a mixed social choice. In this connection, we will require the following conditions (whose labels continue the series started in [1, 2]):

G   *Fraction-like form.* Each fraction-like rate is a number greater than or equal to 0. Their sum is equal to 1.

H   *Fraction-like decomposition.* Let X be a non-empty subset of options with the property that each member of X is unanimously preferred to any alternative outside X. (H1) In that case, the fraction-like rates should vanish outside X. (H2) Here we assume that the individual votes are complete, or alternatively, that each of them is a ranking (possibly truncated or with ties). *If X has the property stated above and no proper subset has the same property,*

then the fraction-like rates of X are all of them positive and they coincide with those that one obtains when the individual votes are restricted to X. (H3) In the complete case, the converse of the latter implication holds too.

I   *Fraction-like rates for single-choice voting.* Assume that each ballot reduces to choosing a single option. In that case, the fraction-like rates coincide with the obtained vote fractions.

In connection with condition I, it is important to notice that single-choice voting results in a very special class of Llull matrices. In fact, if a single-choice ballot for x is interpreted as meaning *nothing else* than a preference for x above any other option, then we get a system of preference scores of the form $v_{xy} = f_x$ for any $y \neq x$, where f_x is the fraction of the vote in favour of the option x.

The mean preference scores ρ_x can always be rescaled so as to satisfy condition G. On the other hand, they are easily seen to satisfy also condition I. In fact, having $v_{xy} = f_x$ for any $y \neq x$ certainly implies $\rho_x = f_x$. However, they definitely do not satisfy the decomposition condition H. For instance, for $A = \{a, b, c\}$ with $v_{ab} = v_{ac} = v_{bc} = 1$ (which implies $v_{ba} = v_{ca} = v_{cb} = 0$) we are in the situation considered by that condition, with $X = \{a\}$, but we have $\rho_b = 1/2 > 0$.

In connection with condition H, it is quite natural to consider another celebrated method that takes as rates the components of a non-negative (right) eigenvector of the Llull matrix (see $[7 \colon \S 2]$ and the references therein). For definiteness, here it is convenient to put $v_{xx} = {}^1\!/_2$ for any x. In the situation considered by condition H, namely $v_{xy} = 1$, and therefore $v_{yx} = 0$, for any $x \in X$ and $y \notin X$, the Llull matrix has a non-negative eigenvector (u_x) of the following form: $u_x = 0$ for $x \notin X$; $(u_x \mid x \in X)$ is a non-negative eigenvector of the submatrix $(v_{xy} \mid x, y \in X)$. This is quite in agreement with the requirements of condition H. However, one can see that such a rating method cannot be continuous in the dependence of the rates with respect to the entries of the Llull matrix. For instance, one can easily check that, for $0 < \epsilon < 1$, the matrix

$$\begin{pmatrix} {}^1\!/_2 & 1 - \epsilon & 1 - \epsilon \\ \epsilon & {}^1\!/_2 & {}^1\!/_2 \\ \epsilon & {}^1\!/_2 & {}^1\!/_2 \end{pmatrix}$$

has a unique non-negative eigenvector (unique up to multiplication by a positive number), namely $\left(8(1 - \epsilon)(1 + \sqrt{1 + 32\epsilon - 32\epsilon^2}\,)^{-1},\, 1,\, 1 \right)$, whose limit as $\epsilon \downarrow 0$ is $(4, 1, 1)$, and not $(1, 0, 0)$, as required by condition H.

In contrast, we will see that Zermelo's method of strengths has the virtue of achieving at the same time both condition H and a property of continuity of the rates with respect to the entries of the Llull matrix.

1.4 In connection with these matters, we need to introduce a qualitative notion of priority that also bears relation to the CLC projection procedure. In order to define it we will make use of the indirect scores v_{xy}^*: given x and y, one considers all possible paths $x_0 x_1 \ldots x_n$ going from $x_0 = x$ to $x_n = y$; every such path is associated with the score of its weakest link, i.e. the smallest value of $v_{x_i x_{i+1}}$; finally, v_{xy}^* is defined as the maximum value of this associated score over all paths from x to y. In other words,

$$v_{xy}^* \quad = \quad \max_{\substack{x_0 = x \\ x_n = y}} \quad \min_{\substack{i \geq 0 \\ i < n}} \quad v_{x_i x_{i+1}}, \tag{7}$$

where the max operator considers all possible paths from x to y, and the min operator considers all the links of a particular path.

By the definition of v_{xy}^*, the inequality $v_{xy}^* > 0$ clearly defines a transitive relation. In the following we will denote it by the symbol $\unrhd$. Thus,

$$x \unrhd y \iff v_{xy}^* > 0. \tag{8}$$

Associated with it, it is interesting to consider also the following derived relations, which keep the property of transitivity and are respectively symmetric and asymmetric:

$$x \equiv y \iff v_{xy}^* > 0 \text{ and } v_{yx}^* > 0, \tag{9}$$

$$x \rhd y \iff v_{xy}^* > 0 \text{ and } v_{yx}^* = 0. \tag{10}$$

Therefore, $\equiv$ is an equivalence relation and $\rhd$ is a partial order. In the following, the situation where $x \rhd y$ will be expressed by saying that x **dominates** y.

The equivalence classes of A by $\equiv$ are called the **irreducible components** of A (for V). If there is only one of them, namely A itself, then one says that the matrix V is irreducible. So, V is irreducible if and only if $v_{xy}^* > 0$ for any $x, y \in A$. It is not difficult to see that this property is equivalent to the following one formulated in terms of the direct scores only: there is no splitting of A into two classes X and Y so that $v_{yx} = 0$ for any $x \in X$ and $y \in Y$; in other words, there is no ordering of A for which the matrix V takes the form

$$\begin{pmatrix} \mathsf{V}_{XX} & \mathsf{V}_{XY} \\ \mathsf{O} & \mathsf{V}_{YY} \end{pmatrix}, \tag{11}$$

where V_{XX} and V_{YY} are square matrices and O is a zero matrix. Besides, a subset $X \subseteq A$ is an irreducible component if and only if X is maximal, in the sense of set inclusion, for the property of V_{XX} being irreducible. On the other hand, it also happens that the relations $\unrhd$ and $\rhd$ are compatible with the equivalence relation $\equiv$, i.e. if $x \equiv \bar{x}$ and $y \equiv \bar{y}$ then $x \unrhd y$ implies $\bar{x} \unrhd \bar{y}$, and analogously $x \rhd y$ implies $\bar{x} \rhd \bar{y}$. As a consequence, the relations $\unrhd$ and $\rhd$ can be applied also to the irreducible components of A for V. In the following we will be interested in the case where V is irreducible, or more generally, when there is a **top dominant irreducible component**, i.e. an irreducible component which dominates any other.

2 Zermelo's method of strengths

The Llull matrix of a vote among V voters can be viewed as corresponding to a tournament between the members of A. In fact, it is as if x and y had played $T_{xy} = t_{xy}V$ matches (the number of voters who made a comparison between x and y, even if this comparison resulted in a tie) and $V_{xy} = v_{xy}V$ of these matches had been won by x, whereas the other $V_{yx} = v_{yx}V$ had been won by y (one tied match is counted as half a match in favour of x plus half a match in favour of y). For such a scenario, Ernst Zermelo [13] devised in 1929 a rating method which turns out to be quite suitable to convert our rank-like rates into fraction-like ones. This method was rediscovered later on by other autors (see [12, 7] and the references therein).

Zermelo's method is based upon a probabilistic model for the outcome of a match between two items x and y. This model assumes that such a match is won by x with probability $\varphi_x/(\varphi_x + \varphi_y)$ whereas it is won by y with probability $\varphi_y/(\varphi_x + \varphi_y)$, where φ_x is a non-negative parameter associated with each player x, usually referred to as its **strength**. If all matches are independent events, the probability of obtaining a particular system of values for the scores (V_{xy}) is given by

$$P = \prod_{\{x,y\}} \binom{T_{xy}}{V_{xy}} \left(\frac{\varphi_x}{\varphi_x + \varphi_y}\right)^{V_{xy}} \left(\frac{\varphi_y}{\varphi_x + \varphi_y}\right)^{V_{yx}}, \tag{12}$$

where the product runs through all unordered pairs $\{x, y\} \subseteq A$ with $x \neq y$. Notice that P depends only on the strength ratios; in other words, multiplying all the strengths by the same value has no effect on the result. On account of this, we will normalize the strengths by requiring their sum to be equal to 1. In order to include certain extreme cases, one must allow for some of the strengths to vanish. However, this may conflict with P being well defined, since it could lead to indeterminacies of the type $0/0$ or 0^0.

So, one should be careful in connection with vanishing strengths. With all this in mind, for the moment we will let the strengths vary in the following set:

$$Q = \{\, \varphi \in \mathbb{R}^A \mid \varphi_x > 0 \text{ for all } x \in A, \sum_{x \in A} \varphi_x = 1 \,\}. \tag{13}$$

Together with this set, in the following we will consider also its closure $\overline{Q}$, which includes vanishing strengths, and its boundary $\partial Q = \overline{Q} \setminus Q$. As it will be seen below, Zermelo's method corresponds to a maximum likelihood estimate of the parameters φ_x from a given set of actual values of V_{xy} (and of $T_{xy} = V_{xy} + V_{yx}$). In other words, given the values of V_{xy}, one looks for the values of φ_x which maximize the probability P.

The hypothesis of independence which lies behind formula (12) is certainly not satisfied by the binary comparisons which arise out of preferential voting. However, it turns out that the same estimates of the parameters φ_x arise from a related model where the voters are assumed to express complete definite rankings ('definite' means here 'without ties'). Both Zermelo's binary model and the ranking model that we are about to introduce can be viewed as special cases of a more general model, proposed in 1959 by Robert Duncan Luce, which considers the outcome of making a choice out of multiple options [10]. According to Luce's "choice axiom", the probabilities of two different choices x and y are in a ratio which does not depend on which other options are present. As a consequence, it follows that every option x can be associated a number φ_x so that the probability of choosing x out of a set $X \ni x$ is given by $\varphi_x/(\sum_{y \in X} \varphi_y)$. Obviously, Zermelo's model corresponds to considering binary choices only. However, Luce's model also allows to associate every complete definite ranking with a certain probability. In fact, such a ranking can be viewed as the result of first choosing the winner out of the whole set A, then choosing the best of the remainder, and so on. If these successive choices are assumed to be independent events, then one can easily figure out the corresponding probability. Furthermore, one can see that these probabilities make the expected rank of x equal to $E(r_x) = N - \sum_{y \neq x} \varphi_x/(\varphi_x + \varphi_y)$. By equating these values to the mean ranks given by equations (5–6), namely $\bar{r}_x = N - \sum_{y \neq x} v_{xy}$ —so using the so-called method of moments— one obtains exactly the same equations for the estimated values of the parameters φ_x as in the method of maximum likelihood, namely equations (15) below. Notice also that, in accordance with Luce's theory of choice, the normalization condition $\sum_{x \in A} \varphi_x = 1$ allows to view φ_x as the first-choice probability of x (among non-abstainers). Anyway, i.e. independenty of the reasons behind them, the resulting values of φ_x will be seen to have good properties for transforming our projected scores into fraction-like rates.

In the following we take the point of view of maximum likelihood. So, given the values of V_{xy}, we will look for the values of φ_x which maximize the probability P. Since V_{xy} and $T_{xy} = V_{xy} + V_{yx}$ are now fixed, this is equivalent to maximizing the following function of the φ_x:

$$F(\varphi) = \prod_{\{x,y\}} \frac{\varphi_x^{v_{xy}} \varphi_y^{v_{yx}}}{(\varphi_x + \varphi_y)^{t_{xy}}}, \qquad (14)$$

(recall that $v_{xy} = V_{xy}/V$ and $t_{xy} = T_{xy}/V$ where V is a positive constant greater than or equal to any of the turnouts T_{xy}; going from (12) to (14) involves taking the power of exponent $1/V$ and disregarding a fixed multiplicative constant). The function F is certainly smooth on Q. Besides, it is clearly bounded from above, since it is a product of several factors less than or equal to 1. However, generally speaking F needs not to achieve a maximum in Q, because this set is not compact. In the present situation, the only general fact that one can guarantee in this connection is the existence of maximizing sequences, i.e. sequences φ^n in Q with the property that $F(\varphi^n)$ converges to the lowest upper bound $\overline{F} = \sup\{F(\psi) \,|\, \psi \in Q\}$.

The next theorems collect the basic results that we need about Zermelo's method.

Theorem 2.1 (Zermelo, 1929 [*13*]; see also [*5, 7*])**.** *If* V *is irreducible, then:*

(a) *There is a unique* $\varphi \in Q$ *which maximizes* F *on* Q.

(b) φ *is the solution of the following system of equations:*

$$\sum_{y \neq x} t_{xy} \frac{\varphi_x}{\varphi_x + \varphi_y} = \sum_{y \neq x} v_{xy}, \qquad (15)$$

$$\sum_x \varphi_x = 1, \qquad (16)$$

where (15) *contains one equation for every* x.

(c) φ *is an infinitely differentiable function of the scores* v_{xy} *as long as they keep satisfying the hypothesis of irreducibility.*

Proof. Let us begin by noticing that the hypothesis of irreducibility entails that F can be extended to a continuous function on $\overline{Q}$ by putting $F(\psi) = 0$ for $\psi \in \partial Q$. In order to prove this claim we must show that $F(\psi^n) \to 0$ whenever ψ^n converges to a point $\psi \in \partial Q$. Let us consider the following sets associated with ψ: $X = \{x \,|\, \psi_x > 0\}$ and $Y = \{y \,|\, \psi_y = 0\}$. The second one is not empty since we are assuming $\psi \in \partial Q$, whereas the first one is not empty because the strengths add up to the positive value 1. Now, for any $x \in X$ and $y \in Y$, $F(\psi^n)$ contains a factor of the form $(\psi_y^n)^{v_{yx}}$, which tends

to zero as soon as $v_{yx} > 0$. So, the only way for $F(\psi^n)$ not to approach zero would be $\mathsf{V}_{YX} = \mathsf{O}$, in contradiction with the irreducibility of V.

After such an extension, F is a continuous function on the compact set $\overline{Q}$. So, there exists φ which maximizes F on $\overline{Q}$. However, since $F(\psi)$ vanishes on ∂Q whereas it is strictly positive for $\psi \in Q$, any maximizer φ must belong to Q. This establishes the existence part of (a).

Since F is constant on every ray from the origin, maximizing it on Q amounts to the same thing as maximizing it on the positive orthant $\mathbb{R}_+^A$. On the other hand, maximizing F is certainly equivalent to maximizing $\log F$. Now, a maximizer of $\log F$ on $\mathbb{R}_+^A$ must satisfy the differential conditions

$$\frac{\partial \log F(\varphi)}{\partial \varphi_x} = \sum_{y \neq x} \left(\frac{v_{xy}}{\varphi_x} - \frac{t_{xy}}{\varphi_x + \varphi_y} \right) = 0, \tag{17}$$

where x varies over A. Multiplying each of these equations by the corresponding φ_x results in the system of equations (15). That system contains N equations for the N variables φ_x ($x \in A$); however, it is redundant: by using the fact that $v_{xy} + v_{yx} = t_{xy}$, one easily sees that adding up all of the equations in (15) results in a tautology. That is why one can supplement that system with equation (16), which selects the maximizer in Q.

Let us see now that the maximizer is unique. Instead of following the interesting proof given by Zermelo, here we will prefer to follow [7], which will have the advantage of preparing matters for part (c). More specifically, the uniqueness will be obtained by seeing that any critical point of $\log F$ as a function on Q, i.e. any solution of (15–16), is a strict local maximum; this implies that there is only one critical point, because otherwise one should have other kinds of critical points [4 : §VI.6] (we are invoking the so-called mountain pass theorem; here we are using the fact that $\log F$ tends to $-\infty$ as φ approaches ∂Q). In order to study the character of a critical point we will look at the second derivatives of $\log F$ with respect to φ. By differentiating (17), one obtains that

$$\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x{}^2} = -\sum_{y \neq x} \left(\frac{v_{xy}}{\varphi_x^2} - \frac{t_{xy}}{(\varphi_x + \varphi_y)^2} \right), \tag{18}$$

$$\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x \, \partial \varphi_y} = \frac{t_{xy}}{(\varphi_x + \varphi_y)^2}, \qquad \text{for } x \neq y. \tag{19}$$

On the other hand, when φ is a critical point, equation (15) transforms (18) into the following expression:

$$\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x{}^2} = -\sum_{y \neq x} \frac{t_{xy}}{(\varphi_x + \varphi_y)^2} \frac{\varphi_y}{\varphi_x}. \tag{20}$$

So, the Hessian bilinear form is as follows:

$$
\begin{aligned}
\sum_{x,y} \left(\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x \, \partial \varphi_y} \right) \psi_x \, \psi_y &= -\sum_{x,y \neq x} \frac{t_{xy}}{(\varphi_x + \varphi_y)^2} \left(\frac{\varphi_y}{\varphi_x} \psi_x^2 - \psi_x \psi_y \right) \\
&= -\sum_{x,y \neq x} \frac{t_{xy}}{(\varphi_x + \varphi_y)^2 \, \varphi_x \varphi_y} \left(\varphi_y^2 \psi_x^2 - \varphi_x \varphi_y \psi_x \psi_y \right) \qquad (21) \\
&= -\sum_{\{x,y\}} \frac{t_{xy}}{(\varphi_x + \varphi_y)^2 \, \varphi_x \varphi_y} \left(\varphi_y \psi_x - \varphi_x \psi_y \right)^2 ,
\end{aligned}
$$

where the last sum runs through all unordered pairs $\{x, y\} \subseteq A$ with $x \neq y$. The last expression is non-positive and it vanishes if and only if $\psi_x / \varphi_x = \psi_y / \varphi_y$ for any $x, y \in A$ (the "only if" part is immediate when $t_{xy} > 0$; for arbitrary x and y, the hypothesis of irreducibility allows to connect them through a path $x_0 x_1 \ldots x_n$ ($x_0 = x$, $x_n = y$) with the property that $t_{x_i x_{i+1}} \geq v_{x_i x_{i+1}} > 0$ for any i, so that one gets $\psi_x / \varphi_x = \psi_{x_1} / \varphi_{x_1} = \cdots = \psi_y / \varphi_y$). So, the vanishing of (21) happens if and only if $\psi = \lambda \varphi$ for some scalar λ. However, when ψ is restricted to variations so as to stay in Q, i.e. to vectors $\psi \in \mathbb{R}^A$ satisfying $\sum_x \psi_x = 0$, the case $\psi = \lambda \varphi$ reduces to $\lambda = 0$ and therefore $\psi = 0$ (since $\sum_x \varphi_x$ is positive). So, the Hessian is negative definite when restricted to such variations. This ensures that φ is a strict local maximum of $\log F$ as a function on Q. In fact, one easily arrives at such a conclusion when Taylor's formula is used to analyse the behaviour of $\log F(\varphi + \psi)$ for small ψ satisfying $\sum_x \psi_x = 0$.

Finally, let us consider the dependence of $\varphi \in Q$ on the matrix V. To begin with, we notice that the set $\mathcal{I}$ of irreducible matrices is open since it is a finite intersection of open sets, namely one open set for each splitting of A into two sets X and Y. The dependence of $\varphi \in Q$ on V is due to the presence of v_{xy} and $t_{xy} = v_{xy} + v_{yx}$ in the equations (15–16) which determine φ. However, we are not in the standard setting of the implicit function theorem, since we are dealing with a system of $N+1$ equations whilst φ varies in a space of dimension $N-1$. In order to place oneself in a standard setting, it is convenient here to replace the condition of normalization $\sum_x \varphi_x = 1$ by the alternative one $\varphi_a = 1$, where a is a fixed element of A. This change of normalization corresponds to mapping Q to $U = \{ \varphi \in \mathbb{R}^A \,|\, \varphi_x > 0 \text{ for all } x \in A,\ \varphi_a = 1 \}$ by means of the diffeomorphism $g : \varphi \mapsto \varphi / \varphi_a$, which has the property that $F(g(\varphi)) = F(\varphi)$. By an argument of the same kind as that used at the end of the preceding paragraph, one sees that the Hessian bilinear form of $\log F$ is negative definite when restricted to variations so as to stay in U. Therefore, if we take as coordinates on U the φ_x with $x \in A \setminus \{a\} =: A'$, the function F restricted

to U has the property that the matrix $(\partial^2 \log F(\varphi)/\partial\varphi_x \partial\varphi_y \mid x, y \in A')$ is negative definite and therefore invertible, which entails that the system of equations $(\partial \log F(\varphi, \mathsf{V})/\partial\varphi_x = 0 \mid x \in A')$ determines $\varphi \in U$ as a smooth function of $\mathsf{V} \in \mathcal{I}$. $\qquad\qquad\square$

The next theorem is the core result for the compatibility between the decomposition condition H and the continuity of the rates with respect to the data. Let us recall that a maximizing sequence means a sequence $\varphi^n \in Q$ such that $F(\varphi^n)$ approaches the lowest upper bound of F on Q.

Theorem 2.2 (Statements (a) and (b) are proved in [13]; results related to (c) are contained in [3]). *Assume that there exists a top dominant irreducible component X. In this case:*

(a) *There is a unique $\varphi \in \overline{Q}$ such that any maximizing sequence converges to φ.*

(b) *$\varphi_{A \setminus X} = 0$, whereas φ_X coincides with the solution of a system analogous to (15–16) where x and y vary only within X.*

(c) *φ is a continuous function of the scores v_{xy} as long as they keep satisfying the hypotheses of the present theorem.*

Proof. The definition of the lowest upper bound immediately implies the existence of maximizing sequences. On the other hand, the compactness of $\overline{Q}$ guarantees that any maximizing sequence has a subsequence which converges in $\overline{Q}$. Let φ^n and φ denote respectively one of such convergent maximizing sequences and its limit. In the following we will see that φ must be the unique point specified in statement (b). This entails that any maximizing sequence converges itself to φ (without extracting a subsequence).

So, our aim is now statement (b). From now on we will write $Y = A \setminus X$, and a general element of $\mathbb{R}_+^A$ will be denoted by ψ. For convenience, in this part of the proof we will replace the condition $\sum_x \psi_x = 1$ by $\sum_x \psi_x \leq 1$ (and similarly for φ^n and φ); since $F(\lambda\psi) = F(\psi)$ for any $\lambda > 0$, the properties that we will obtain will be easily translated from $\widehat{Q} = \{ \psi \in \mathbb{R}^A \mid \psi_x > 0 \text{ for all } x \in A, \ \sum_{x \in A} \psi_x \leq 1 \}$ to Q. On the other hand, it will also be convenient to consider first the case where Y is also an irreducible component. In such a case, it is interesting to rewrite $F(\psi)$ as a product of three factors:
$$F(\psi) = F_{XX}(\psi_X)\, F_{YY}(\psi_Y)\, F_{XY}(\psi_X, \psi_Y), \qquad (22)$$
namely:
$$F_{XX}(\psi_X) = \prod_{\{x,\bar{x}\} \subseteq X} \frac{\psi_x^{\,v_{x\bar{x}}} \psi_{\bar{x}}^{\,v_{\bar{x}x}}}{(\psi_x + \psi_{\bar{x}})^{t_{x\bar{x}}}}, \qquad (23)$$

$$F_{YY}(\psi_Y) \;=\; \prod_{\{y,\bar{y}\}\subseteq Y} \frac{\psi_y^{\,v_{y\bar{y}}}\,\psi_{\bar{y}}^{\,v_{\bar{y}y}}}{(\psi_y + \psi_{\bar{y}})^{t_{y\bar{y}}}}, \tag{24}$$

$$F_{XY}(\psi_X, \psi_Y) \;=\; \prod_{\substack{x\in X \\ y\in Y}} \left(\frac{\psi_x}{\psi_x + \psi_y}\right)^{v_{xy}}, \tag{25}$$

where we used that $v_{yx} = 0$ and $t_{xy} = v_{xy}$. Now, let us look at the effect of replacing ψ_Y by $\lambda\psi_Y$ without varying ψ_X. The values of F_{XX} and F_{YY} remain unchanged, but that of F_{XY} varies in the following way:

$$\frac{F_{XY}(\psi_X, \lambda\psi_Y)}{F_{XY}(\psi_X, \psi_Y)} \;=\; \prod_{\substack{x\in X \\ y\in Y}} \left(\frac{\psi_x + \psi_y}{\psi_x + \lambda\psi_y}\right)^{v_{xy}}. \tag{26}$$

In particular, for $0 < \lambda < 1$ each of the factors of the right-hand side of (26) is greater than 1. This remark leads to the following argument. First, we can see that $\varphi_y^n/\varphi_x^n \to 0$ for any $x \in X$ and $y \in Y$ such that $v_{xy} > 0$ (such pairs xy exist because of the hypothesis that X dominates Y). Otherwise, the preceding remark entails that the sequence $\widetilde{\varphi}^n = (\varphi_X^n, \lambda\varphi_Y^n)$ with $0 < \lambda < 1$ would satisfy $F(\widetilde{\varphi}^n) > KF(\varphi^n)$ for some $K > 1$ and infinitely many n, in contradiction with the hypothesis that φ^n was a maximizing sequence. On the other hand, we see also that $F_{XY}(\varphi^n)$ approaches its lowest upper bound, namely 1. Having achieved such a property, the problem of maximizing F reduces to separately maximizing F_{XX} and F_{YY}, which is solved by Theorem 2.1. For the moment we are dealing with relative strengths only, i.e. without any normalizing condition like (16). So, we see that F_{YY} gets optimized when each of the ratios $\varphi_y^n/\varphi_{\bar{y}}^n$ $(y, \bar{y} \in Y)$ approaches the homologous one for the unique maximizer of F_{YY}, and analogously with F_{XX}. Since these ratios are finite positive quantities, the statement that $\varphi_y^n/\varphi_x^n \to 0$ becomes extended to any $x \in X$ and $y \in Y$ whatsoever (since one can write $\varphi_y^n/\varphi_x^n = (\varphi_y^n/\varphi_{\bar{y}}^n)\times(\varphi_{\bar{y}}^n/\varphi_{\bar{x}}^n)\times(\varphi_{\bar{x}}^n/\varphi_x^n)$ with $v_{\bar{x}\bar{y}} > 0$). Let us recover now the condition $\sum_{x\in A}\varphi_x^n = 1$. The preceding facts imply that $\varphi_Y^n \to 0$, whereas φ_X^n converges to the unique maximizer of F_{XX}. This establishes (b) as well as the uniqueness part of (a).

The general case where Y decomposes into several irreducible components, all of them dominated by X, can be taken care of by induction over the different irreducible components of A. At each step, one deals with an irreducible component Z with the property of being minimal, in the sense of the dominance relation $\rhd$, among those which are still pending. By means of an argument analogous to that of the preceding paragraph, one sees that: (i) $\varphi_z^n/\varphi_x^n \to 0$ for any $z \in Z$ and x such that $x \rhd z$ with

$v_{xz} > 0$; (ii) the ratios $\varphi_z^n/\varphi_{\bar{z}}^n$ ($z, \bar{z} \in Z$) approach the homologous ones for the unique maximizer of F_{ZZ}; and (iii) φ_R^n is a maximizing sequence for F_{RR}, where R denotes the union of the pending components, Z excluded. Once this induction process has been completed, one can combine its partial results to show that $\varphi_z^n/\varphi_x^n \to 0$ for any $x \in X$ and $z \notin X$ (it suffices to consider a path $x_0 x_1 \ldots x_n$ from $x_0 \in X$ to $x_n = z$ with the property that $v_{x_i x_{i+1}} > 0$ for any i and to notice that each of the factors $\varphi_{x_{i+1}}^n/\varphi_{x_i}^n$ remains bounded while at least one of them tends to zero). As above, one concludes that $\varphi_{A \setminus X}^n \to 0$, whereas φ_X^n converges to the unique maximizer of F_{XX}.

The two following remarks will be useful in the proof of part (c): (1) According to the proof above, φ_X is determined (up to a multiplicative constant) by equations (15) with x and y varying only within X:

$$\mathcal{F}_x(\varphi_X, \mathsf{V}) := \sum_{\substack{y \in X \\ y \neq x}} t_{xy} \frac{\varphi_x}{\varphi_x + \varphi_y} - \sum_{\substack{y \in X \\ y \neq x}} v_{xy} = 0, \qquad \forall x \in X. \quad (27)$$

However, since $y \in A \setminus X$ implies on the one hand $\varphi_y = 0$ and on the other hand $t_{xy} = v_{xy}$, each of the preceding equations is equivalent to a similar one where y varies over the whole of $A \setminus \{x\}$:

$$\mathcal{F}_x'(\varphi, \mathsf{V}) := \sum_{\substack{y \in A \\ y \neq x}} t_{xy} \frac{\varphi_x}{\varphi_x + \varphi_y} - \sum_{\substack{y \in A \\ y \neq x}} v_{xy} = 0, \qquad \forall x \in X. \quad (28)$$

(2) Also, it is interesting to see the result of adding up the equations (28) for all x in some subset W of X. Using the fact that $v_{xy} + v_{yx} = t_{xy}$, one sees that such an addition results in the following equality:

$$\sum_{\substack{x \in W \\ y \notin W}} t_{xy} \frac{\varphi_x}{\varphi_x + \varphi_y} - \sum_{\substack{x \in W \\ y \notin W}} v_{xy} = 0, \qquad \forall W \subseteq X. \quad (29)$$

Let us proceed now with the proof of (c). In the following, V and $\widetilde{\mathsf{V}}$ denote respectively a fixed matrix satisfying the hypotheses of the theorem and a slight perturbation of it. In the following we systematically use a tilde to distinguish between homologous objects associated respectively with V and $\widetilde{\mathsf{V}}$; in particular, such a notation will be used in connection with the labels of certain equations. Our aim is to show that $\widetilde{\varphi}$ approaches φ as $\widetilde{\mathsf{V}}$ approaches V. In this connection we will use the little-o and big-O notations made popular by Edmund Landau (who by the way is the author of a paper on the rating of chess players, namely [8], which inspired Zermelo's work). This

notation refers here to functions of $\widetilde{\mathsf{V}}$ and their behaviour as $\widetilde{\mathsf{V}}$ approaches V; if f and g are two such functions, $f = o(g)$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\widetilde{\mathsf{V}} - \mathsf{V}\| \le \delta$ implies $\|f(\widetilde{\mathsf{V}})\| \le \epsilon \|g(\widetilde{\mathsf{V}})\|$; on the other hand, $f = O(g)$ means that there exist M and $\delta > 0$ such that $\|\widetilde{\mathsf{V}} - \mathsf{V}\| \le \delta$ implies $\|f(\widetilde{\mathsf{V}})\| \le M \|g(\widetilde{\mathsf{V}})\|$.

Obviously, if $\widetilde{\mathsf{V}}$ is near enough to V then $v_{xy} > 0$ implies $\widetilde{v}_{xy} > 0$. As a consequence, $x \unrhd y$ implies $x \mathbin{\widetilde{\unrhd}} y$. In particular, the irreducibility of V_{XX} entails that $\widetilde{\mathsf{V}}_{XX}$ is also irreducible. Therefore, X is entirely contained in some irreducible component $\widetilde{X}$ of A for $\widetilde{\mathsf{V}}$. Besides, $\widetilde{X}$ is a top dominant irreducible component for $\widetilde{\mathsf{V}}$; in fact, we have the following chain of implications for $x \in X \subseteq \widetilde{X}$: $y \notin \widetilde{X} \Rightarrow y \notin X \Rightarrow x \rhd y \Rightarrow x \mathbin{\widetilde{\unrhd}} y \Rightarrow x \mathbin{\widetilde{\rhd}} y$, where we have used successively: the inclusion $X \subseteq \widetilde{X}$, the hypothesis that X is top dominant for V, the fact that $\widetilde{\mathsf{V}}$ is near enough to V, and the hypothesis that y does not belong to the irreducible component $\widetilde{X}$. Now, according to part (b) and remark (1) from p. 14–14, φ_X and $\widetilde{\varphi}_{\widetilde{X}}$ are determined respectively by the systems (27) and $(\widetilde{27})$, or equivalently by (28) and $(\widetilde{28})$, whereas $\varphi_{A \setminus X}$ and $\widetilde{\varphi}_{A \setminus \widetilde{X}}$ are both of them equal to zero. So we must show that $\widetilde{\varphi}_y = o(1)$ for any $y \in \widetilde{X} \setminus X$, and that $\widetilde{\varphi}_x - \varphi_x = o(1)$ for any $x \in X$. The proof is organized in three main steps.

Step (1). $\widetilde{\varphi}_y = O(\widetilde{\varphi}_x)$ *whenever* $v_{xy} > 0$. For the moment, we assume $\widetilde{\mathsf{V}}$ fixed (near enough to V so that $\widetilde{v}_{xy} > 0$) and $x, y \in \widetilde{X}$. Under these hypotheses one can argue as follows: Since $\widetilde{\varphi}_{\widetilde{X}}$ maximizes $\widetilde{F}_{\widetilde{X}\widetilde{X}}$, the corresponding value of $\widetilde{F}_{\widetilde{X}\widetilde{X}}$ can be bounded from below by any particular value of the same function. On the other hand, we can bound it from above by the factor $\widetilde{\varphi}_x / (\widetilde{\varphi}_x + \widetilde{\varphi}_y)^{\widetilde{v}_{xy}}$. So, we can write

$$\left(\frac{1}{2}\right)^{N(N-1)} \le \left(\frac{1}{2}\right)^{\sum\limits_{p,q \in \widetilde{X}} \widetilde{t}_{pq}} = \widetilde{F}_{\widetilde{X}\widetilde{X}}(\psi) \le \widetilde{F}_{\widetilde{X}\widetilde{X}}(\widetilde{\varphi}_{\widetilde{X}}) \le \left(\frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y}\right)^{\widetilde{v}_{xy}}, \quad (30)$$

where ψ has been taken so that ψ_q has the same value for all $q \in \widetilde{X}$ (and it vanishes for $q \notin \widetilde{X}$). The preceding inequality entails that

$$\widetilde{\varphi}_y \le \left(2^{N(N-1)/\widetilde{v}_{xy}} - 1\right) \widetilde{\varphi}_x. \quad (31)$$

Now, this inequality holds not only for $x, y \in \widetilde{X}$, but it is also trivially true for $y \notin \widetilde{X}$, since then one has $\widetilde{\varphi}_y = 0$. On the other hand, the case $y \in \widetilde{X}$, $x \notin \widetilde{X}$ is not possible at all, because the hypothesis that $\widetilde{v}_{xy} > 0$ would then contradict the fact that $\widetilde{X}$ is a top dominant irreducible component. Finally, we let $\widetilde{\mathsf{V}}$ vary towards V. The desired result is a consequence of (31) since $\widetilde{v}_{xy}$ approaches $v_{xy} > 0$.

Step (2). $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$ *for any* $x \in X$ *and* $y \notin X$. Again, we will consider first the special case where $v_{xy} > 0$. In this case the result is easily obtained as a consequence of the equality $(\widetilde{29})$ for $W = X$:

$$\sum_{\substack{x \in X \\ y \notin X}} \widetilde{t}_{xy} \, \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} - \sum_{\substack{x \in X \\ y \notin X}} \widetilde{v}_{xy} = 0. \tag{32}$$

In fact, this equality implies that

$$\sum_{\substack{x \in X \\ y \notin X}} \widetilde{t}_{xy} \left(1 - \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} \right) = \sum_{\substack{x \in X \\ y \notin X}} \widetilde{v}_{yx}. \tag{33}$$

Now, it is clear that the right-hand side of this equation is $o(1)$ and that each of the terms of the left-hand side is positive. Since $\widetilde{t}_{xy} - v_{xy} = \widetilde{t}_{xy} - t_{xy} = o(1)$, the hypothesis that $v_{xy} > 0$ allows to conclude that $\widetilde{\varphi}_x / (\widetilde{\varphi}_x + \widetilde{\varphi}_y)$ approaches 1, or equivalently, $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$. Let us consider now the case of any $x \in X$ and $y \notin X$. Since X is top dominant, we know that there exists a path $x_0 x_1 \ldots x_n$ from $x_0 = x$ to $x_n = y$ such that $v_{x_i x_{i+1}} > 0$ for all i. According to step (1) we have $\widetilde{\varphi}_{x_{i+1}} = O(\widetilde{\varphi}_{x_i})$. On the other hand, there must be some j such that $x_j \in X$ but $x_{j+1} \notin X$, which has been seen to imply that $\widetilde{\varphi}_{x_{j+1}} = o(\widetilde{\varphi}_{x_j})$. By combining these facts one obtains the desired result.

Step (3). $\widetilde{\varphi}_x - \varphi_x = o(1)$ *for any* $x \in X$. Consider the equations $(\widetilde{28})$ for $x \in X$ and split the sums in two parts depending on whether $y \in X$ or $y \notin X$:

$$\sum_{\substack{y \in X \\ y \neq x}} \widetilde{t}_{xy} \, \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} - \sum_{\substack{y \in X \\ y \neq x}} \widetilde{v}_{xy} = \sum_{y \notin X} \left(\widetilde{v}_{xy} - \widetilde{t}_{xy} \, \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} \right). \tag{34}$$

The last sum is $o(1)$ since step (2) ensures that $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$ and we also know that $\widetilde{t}_{xy} - \widetilde{v}_{xy} = \widetilde{v}_{yx} = o(1)$ (because $x \in X$ and $y \notin X$). So $\widetilde{\varphi}$ satisfies a system of the following form, where x and y vary only within X and $\widetilde{w}_{xy}$ is a slight modification of $\widetilde{v}_{xy}$ which absorbs the right-hand side of (34):

$$\mathcal{G}_x(\widetilde{\varphi}_X, \widetilde{\mathsf{V}}, \widetilde{\mathsf{W}}) := \sum_{\substack{y \in X \\ y \neq x}} \widetilde{t}_{xy} \, \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} - \sum_{\substack{y \in X \\ y \neq x}} \widetilde{w}_{xy} = 0, \quad \forall x \in X. \tag{35}$$

Here, the second argument of $\mathcal{G}$ refers to the dependence on $\widetilde{\mathsf{V}}$ through $\widetilde{t}_{xy}$. We know that $\widetilde{t}_{xy} - t_{xy} = o(1)$ and also that $\widetilde{w}_{xy} - v_{xy} = (\widetilde{w}_{xy} - \widetilde{v}_{xy}) +$

$(\widetilde{v}_{xy} - v_{xy}) = o(1)$. So we are interested in the preceding equation near the point $(\varphi_X, \mathsf{V}, \mathsf{V})$. Now in this point we have $\mathcal{G}(\varphi_X, \mathsf{V}, \mathsf{V}) = \mathcal{F}(\varphi_X, \mathsf{V}) = 0$, as well as $(\partial \mathcal{G}_x / \partial \widetilde{\varphi}_y)(\varphi_X, \mathsf{V}, \mathsf{V}) = (\partial \mathcal{F}_x / \partial \varphi_y)(\varphi_X, \mathsf{V})$. Therefore, the implicit function theorem can be applied similarly as in Theorem 2.1, with the result that $\widetilde{\varphi}_X = \mathcal{H}(\widetilde{\mathsf{V}}, \widetilde{\mathsf{W}})$, where $\mathcal{H}$ is a smooth function which satisfies $\mathcal{H}(\mathsf{V}, \mathsf{V}) = \varphi_X$. In particular, the continuity of $\mathcal{H}$ allows to conclude that $\widetilde{\varphi}_X$ approaches φ_X, since we know that both $\widetilde{\mathsf{V}}$ and $\widetilde{\mathsf{W}}$ approach V.

Finally, by combining the results of steps (2) and (3) one obtains $\widetilde{\varphi}_y = o(1)$ for any $y \notin X$. $\qquad\square$

Remarks

1. The convergence of φ^n to φ is a necessary condition for φ^n being a maximizing sequence, but not a sufficient one. The preceding proof shows that a necessary and sufficient condition is that the ratios $\varphi_y^n / \varphi_z^n$ tend to 0 whenever $y \vartriangleright z$, whereas, if $y \equiv z$, i.e. if y and z belong to the same irreducible component Z, these ratios approach the homologous ones for the unique maximizer of F_{ZZ}.

2. If there is not a dominant component, then the maximizing sequences can have multiple limit points. However, as we will see in the next section, the projected Llull matrices are always in the hypotheses of Theorem 2.2.

3. The non-linear system (15–16) can be solved by the following iterative scheme [*13, 5*]:

$$\sum_{y \neq x} t_{xy} \frac{\varphi_x^{(n+1)}}{\varphi_x^{(n)} + \varphi_y^{(n)}} = \sum_{y \neq x} v_{xy}, \tag{36}$$

$$\sum_x \varphi_x^{(n+1)} = 1, \tag{37}$$

3 CLC structure

3.1 In the following we will be interested in Llull matrices that have the following special structure: There exists a total order ξ on A such that

$$v_{xy} \geq v_{yx}, \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y, \tag{38}$$

$$v_{xz} = \max(v_{xy}, v_{yz}), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z, \tag{39}$$

$$v_{zx} = \min(v_{zy}, v_{yx}), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z, \tag{40}$$

$$0 \leq t_{xz} - t_{x'z} \leq m_{xx'}, \qquad \text{whenever } z \notin \{x, x'\}, \tag{41}$$

where $x \overset{\xi}{\succ} y$ means that x precedes y in the order ξ, and x' denotes the element of A that immediately follows x in the order ξ. In the following we will refer to such a situation as **CLC structure**, and the total order ξ will be called an **admissible order** for the matrix (v_{xy}).

Our interest in the CLC structure derives from the following fact in connection with the CLC projection procedure of [1, 2]:

Theorem 3.1 ([2: Thm. 4.5])**.** *The CLC projection procedure always results in a Llull matrix with CLC structure. Besides, a Llull matrix with CLC structure is invariant by the CLC projection procedure.*

In the following we will also make use of the following facts:

Lemma 3.2. *A Llull matrix with CLC structure satisfies the following inequalities:*

$$v_{xz} \geq v_{yz}, \quad v_{zx} \leq v_{zy}, \qquad whenever \ x \overset{\xi}{\succ} y \ and \ z \notin \{x, y\}, \qquad (42)$$

$$t_{xz} \geq t_{yz}, \quad t_{zx} \geq t_{zy}, \qquad whenever \ x \overset{\xi}{\succ} y \ and \ z \notin \{x, y\}. \qquad (43)$$

Proof. It suffices to combine Theorem 3.1 with [2: Thm. 4.3]. $\square$

Proposition 3.3. *A non-vanishing Llull matrix with CLC structure has a top dominant irreducible component X with the special property that*

$$v_{xy} > 0, \qquad whenever \ \ x \in X \ \ and \ \ y \neq x. \qquad (44)$$

Proof. If $v_{xy} > 0$ for all x, y, then (v_{xy}) is irreducible and we are done. So, let us assume that $v_{xy} = 0$ for some x, y. By (38) and (40), this implies the existence of some p such that $v_{p'p} = 0$. Here we are considering an arbitrary admissible order ξ, which we fix for the rest of the proof. Let a be the first element of A according to this order. We will see that the top dominant component is the set X defined by

$$X = \begin{cases} \{x \in A \mid v_{p'p} > 0 \ \text{for all} \ p \overset{\xi}{\succeq} x\}, & \text{if } v_{p'p} > 0 \text{ for some } p, \\ \{a\}, & \text{if } v_{p'p} = 0 \text{ for any } p. \end{cases}$$

From this definition it immediately follows that having $x \in X$ and $y \notin X$ implies $x \overset{\xi}{\succ} y$. This fact will be used repeatedly in the following.

From the definition, it is also clear that for any $x \in X$ and $y \notin X$ there exists p with $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$ such that $v_{p'p} = 0$. By virtue of (40), it follows that

$$v_{yx} = 0, \qquad whenever \ \ x \in X \ and \ y \notin X. \qquad (45)$$

The claim that X is the top dominant component will be a consequence of the preceding property together with (44), to which we devote the rest of the proof.

Let us begin by seeing that $v_{aa'} > 0$. In fact, according to (38) having $v_{aa'} = 0$ would imply $v_{a'a} = 0$ and therefore $t_{aa'} = 0$; by (43), this would imply the vanishing of the whole matrix (v_{xy}), against one of the assumptions. Now, by virtue of (42) it follows that $v_{ay} > 0$ for all $y \neq a$. This finishes the proof if X consists of a only. In the other cases, one can use (38) and (40) to argue by contradiction that $v_{x\bar{x}} > 0$ for all $x, \bar{x} \in X$. Finally, (42) allows to derive that $v_{xy} > 0$ for all $x \in X$ and $y \notin X$, which completes the proof. $\qquad\square$

Lemma 3.4. *A non-vanishing Llull matrix with CLC structure has the following properties, where ξ is any admissible order, ρ_x are the mean preference scores, and X is he top dominant component:*

(a) $x \overset{\xi}{\succ} y$ *implies* $\rho_x \geq \rho_y$.

(b) $\rho_x = \rho_y$ *if and only if* $v_{xy} = v_{yx}$.

(c) $\rho_x \geq \rho_y$ *implies the inequalities* (38) *and* (42).

(d) $\rho_x > \rho_y$ *if and only if* $v_{xy} > v_{yx}$.

(e) $v_{xy} > v_{yx}$ *implies* $x \overset{\xi}{\succ} y$.

(f) $\rho_x > \rho_y$ *whenever* $x \in X$ *and* $y \notin X$.

Proof. In order to obtain (a–e) it suffices to combine Theorem 3.1 with [2 : Lem. 5.1]. In so doing, one has to bear in mind that the mean preference scores ρ_x are related to the rank-like rates R_x of [2] in the following way: $R_x = N - (N-1)\rho_x$. In order to obtain (f) it suffices to combine (d) with Proposition 3.3. $\qquad\square$

Remark. In the complete case, statement (f) is easily seen to hold without any need for invoking the hypothesis of CLC structure [11 : Thm. 2.5].

3.3 In this paragraph we look at the compatibility between strengths and mean ranks.

In this connection, Zermelo proved that in the complete (and irreducible) case the strengths always order the options in exactly the same way as the mean preference scores [13 : § 4].

In the incomplete case, the strengths and the mean preference scores are easily incompatible with each other. In this connection, the mean preference scores may be considered unfair in that different options may not have equal

opportunities for being compared with the others and raising their mean preference score [*12*]. In contrast, Zermelo's strengths seem fairer since they take into account the turnouts in an appropriate way.

The next theorem shows that in the presence of CLC structure both rating methods are equally fair from a qualitative point of view:

Theorem 3.5. *For a Llull matrix with CLC structure, the associated mean preference scores ρ_x and strengths φ_x have the following compatibility properties:*

(a) $\varphi_x > \varphi_y \implies \rho_x > \rho_y$.

(b) $\rho_x > \rho_y \implies$ *either* $\varphi_x > \varphi_y$ *or* $\varphi_x = \varphi_y = 0$.

Proof. In the following X denotes again the top dominant component of the Llull matrix, whose existence has been established by Proposition 3.3; by Theorem 2.2, we know that $\varphi_x > 0$ if and only if $x \in X$. Let us begin by noticing that both statements of the present theorem hold if $\varphi_y = 0$, that is, if $y \notin X$. In this case statement (b) is trivial, while statement (a) holds because of Lemma 3.4.(f). Consider now the case $\varphi_x = 0$. In this case statement (a) is empty, whereas statement (b) reduces, via its contrapositive, to Lemma 3.4.(f) (with x and y interchanged with each other).

So, from now on we can assume that x and y are both in X, or, on account of Theorem 2.2, that $X = A$. In the following we will make use of the results of §2, according to which the strengths (φ_x) are determined by the condition of maximizing the function (14) under the restriction (16), and that they satisfy the equations (15).

Part (a): It will be proved by seeing that a simultaneous occurrence of the inequalities $\varphi_x > \varphi_y$ and $\rho_x \leq \rho_y$ would entail a contradiction with the fact that φ is the unique maximizer of $F(\varphi)$. More specifically, we will see that one would have $F(\widetilde{\varphi}) \geq F(\varphi)$ where $\widetilde{\varphi}$ is obtained from φ by interchanging the values of φ_x and φ_y, that is

$$\widetilde{\varphi}_z = \begin{cases} \varphi_y, & \text{if } z = x, \\ \varphi_x, & \text{if } z = y, \\ \varphi_z, & \text{otherwise.} \end{cases} \tag{46}$$

In fact, $\widetilde{\varphi}$ differs from φ only in the components associated with x and y,

so that

$$
\begin{aligned}
\frac{F(\widetilde{\varphi})}{F(\varphi)} \;=\; & \left(\frac{\widetilde{\varphi}_x}{\varphi_x}\right)^{v_{xy}} \prod_{z\neq x,y} \left(\frac{\widetilde{\varphi}_x/(\widetilde{\varphi}_x+\varphi_z)}{\varphi_x/(\varphi_x+\varphi_z)}\right)^{v_{xz}} \left(\frac{\varphi_x+\varphi_z}{\widetilde{\varphi}_x+\varphi_z}\right)^{v_{zx}} \\
\times\; & \left(\frac{\widetilde{\varphi}_y}{\varphi_y}\right)^{v_{yx}} \prod_{z\neq x,y} \left(\frac{\widetilde{\varphi}_y/(\widetilde{\varphi}_y+\varphi_z)}{\varphi_y/(\varphi_y+\varphi_z)}\right)^{v_{yz}} \left(\frac{\varphi_y+\varphi_z}{\widetilde{\varphi}_y+\varphi_z}\right)^{v_{zy}}.
\end{aligned}
\tag{47}
$$

More particularly, in the case of (46) this expression becomes

$$
\frac{F(\widetilde{\varphi})}{F(\varphi)} \;=\; \left(\frac{\varphi_y}{\varphi_x}\right)^{v_{xy}-v_{yx}} \prod_{z\neq x,y} \left(\frac{\varphi_y/(\varphi_y+\varphi_z)}{\varphi_x/(\varphi_x+\varphi_z)}\right)^{v_{xz}-v_{yz}} \left(\frac{\varphi_y+\varphi_z}{\varphi_x+\varphi_z}\right)^{v_{zy}-v_{zx}}, \tag{48}
$$

where all of the bases are strictly less than 1, since $\varphi_x > \varphi_y$, and all of the the exponents are non-positive, because of Lemma 3.4.(c). Therefore, the product is greater than or equal to 1, as claimed.

Part (b): Since we are assuming $x, y \in X$, it is a matter of proving that $\rho_x > \rho_y \Rightarrow \varphi_x > \varphi_y$. On the other hand, by making use of the contrapositive of (a), the problem reduces to proving that $\varphi_x = \varphi_y \Rightarrow \rho_x = \rho_y$.

Similarly to above, this implication will be proved by seeing that a simultaneous occurrence of the equality $\varphi_x = \varphi_y =: \omega$ together with the inequality $\rho_x > \rho_y$ (by symmetry it suffices to consider this one) would entail a contradiction with the fact that φ is the unique maximizer of $F(\varphi)$. More specifically, here we will see that one would have $F(\widetilde{\varphi}) > F(\varphi)$ where $\widetilde{\varphi}$ is obtained from φ by slightly increasing φ_x while decreasing φ_y, that is

$$
\widetilde{\varphi}_z = \begin{cases} \omega + \epsilon, & \text{if } z = x, \\ \omega - \epsilon, & \text{if } z = y, \\ \varphi_z, & \text{otherwise.} \end{cases} \tag{49}
$$

This claim will be proved by checking that

$$
\left. \frac{\mathrm{d}}{\mathrm{d}\epsilon} \log \frac{F(\widetilde{\varphi})}{F(\varphi)} \right|_{\epsilon=0} > 0. \tag{50}
$$

In fact, (47) entails that

$$
\begin{aligned}
\log \frac{F(\widetilde{\varphi})}{F(\varphi)} \;=\; & C \,+\, v_{xy} \log \widetilde{\varphi}_x + v_{yx} \log \widetilde{\varphi}_y \\
& + \sum_{z\neq x,y} \left(v_{xz} \log \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x+\varphi_z} + v_{yz} \log \frac{\widetilde{\varphi}_y}{\widetilde{\varphi}_y+\varphi_z} \right) \\
& - \sum_{z\neq x,y} \left(v_{zy} \log(\widetilde{\varphi}_y+\varphi_z) + v_{zx} \log(\widetilde{\varphi}_x+\varphi_z) \right),
\end{aligned}
\tag{51}
$$

where C does not depend on ϵ. Therefore, in view of (49) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon} \log \frac{F(\widetilde{\varphi})}{F(\varphi)} \bigg|_{\epsilon=0} = \ & (v_{xy} - v_{yx}) \frac{1}{\omega} + \sum_{z \neq x,y} (v_{xz} - v_{yz}) \frac{\varphi_z}{\omega(\omega + \varphi_z)} \\
& + \sum_{z \neq x,y} (v_{zy} - v_{zx}) \frac{1}{\omega + \varphi_z}.
\end{aligned}
\tag{52}
$$

Now, according to Lemma 3.4.(b, c), the assumption that $\rho_x > \rho_y$ implies the inequalities $v_{xy} > v_{yx}$, $v_{xz} \geq v_{yz}$ and $v_{zy} \geq v_{zx}$, which result indeed in (50). $\qquad\square$

3.4 It is interesting to notice that a Llull matrix with CLC structure keeps an important part of this structure when passing to the relative scores $q_{xy} = v_{xy}/t_{xy}$:

Proposition 3.6. *Assume that (v_{xy}) has CLC structure. If one has $t_{xy} > 0$ for all x, y, then the relative scores $q_{xy} = v_{xy}/t_{xy}$ have the following properties, where ξ is any admissible order for (v_{xy}):*

$$
q_{xy} \geq q_{yx}, \qquad\qquad\quad \text{whenever } x \overset{\xi}{\succsim} y, \tag{53}
$$

$$
q_{xz} \geq q_{yz}, \quad q_{zx} \leq q_{zy}, \qquad \text{whenever } x \overset{\xi}{\succsim} y \text{ and } z \notin \{x, y\}. \tag{54}
$$

Besides, the top dominant irreducible component X of (v_{xy}) is also top dominant irreducible for (q_{xy}), with the special property that

$$
q_{xy} > 0, \qquad \text{whenever } x \in X \text{ and } y \neq x. \tag{55}
$$

If one has $t_{xy} = 0$ for some x, y, then there exists $Y \subseteq A$ such that

$$
t_{x\bar{x}} > 0, \qquad \text{whenever } x, \bar{x} \notin Y, \tag{56}
$$

$$
v_{yx} = 0, \qquad \text{whenever } y \in Y \text{ and } x \neq y. \tag{57}
$$

Proof. Consider first the case where $t_{xy} > 0$ for all x, y. Clearly, (38) immediately implies (53). On the other hand, (42) implies (54) because of the following chains of implications:

$$
\frac{v_{xz}}{t_{xz}} \geq \frac{v_{yz}}{t_{yz}} \iff \frac{t_{xz}}{v_{xz}} \leq \frac{t_{yz}}{v_{yz}} \iff 1 + \frac{v_{zx}}{v_{xz}} \leq 1 + \frac{v_{zy}}{v_{yz}}, \tag{58}
$$

$$
\frac{v_{zx}}{t_{zx}} \geq \frac{v_{zy}}{t_{zy}} \iff \frac{t_{zx}}{v_{zx}} \leq \frac{t_{zy}}{v_{zy}} \iff 1 + \frac{v_{xz}}{v_{zx}} \leq 1 + \frac{v_{yz}}{v_{zy}}. \tag{59}
$$

The statement about the top dominant irreducible component is also an immediate consequence of the positivity of the turnouts.

If $t_{xy} = 0$ for some x, y, then (43) allows to derive that $t_{pp'} = 0$ for some p. If p_1 is the first element with this property and we set $Y = \{y \in A \mid p_1 \overset{\xi}{\succeq} y\}$, we immediately obtain (56), and (42) together with (39) are easily seen to lead to (57). □

As a consequence, we can see that the mean relative preference scores

$$\sigma_x = \frac{1}{N-1} \sum_{y \neq x} v_{xy}/t_{xy} \tag{60}$$

rank the items in the same way as the original mean preference scores:

Corollary 3.7. *Assume that (v_{xy}) has CLC structure and positive turnouts. In that case, one has $\sigma_x > \sigma_y$ if and only if $\rho_x > \rho_y$.*

Proof. In order to prove the stated equivalence, it suffices to prove the two following implications:

$$\rho_x \geq \rho_y \implies \sigma_x \geq \sigma_y, \tag{61}$$

$$\rho_x > \rho_y \implies \sigma_x > \sigma_y. \tag{62}$$

The implication (61) is easily obtained by combining part (c) of Lemma 3.4 with the chain of implications (58). In order to prove (62) it suffices to notice that, according to part (d) of Lemma 3.4, $\rho_x > \rho_y$ implies $v_{xy} > v_{yx}$ and therefore $v_{xy}/t_{xy} > v_{yx}/t_{yx}$. □

Remark. If $t_{xy} = 0$ for some x, y, the last statement of Proposition 3.6 justifies considering any $y \in Y$ categorically worse than any $x \notin Y$, and restricting the rating to the subset $X = A \setminus Y$, which brings the problem to the case of positive turnouts.

4   The CLC projection followed by the method of strengths

In this section we consider the rating method that is obtained by composing the CLC projection procedure of [1, 2] and Zermelo's method of strengths. That is $\Phi = ZP$, where P denotes the CLC projection mapping $(v_{xy}) \mapsto (v_{xy}^\pi)$ and Z denotes the mapping defined by the method of strengths. We will use the following notations: (v_{xy}) denotes the original Llull matrix,

(v_{xy}^π) denotes the projected one, and (φ_x) denotes the resulting strengths. We will refer to them as the **fraction-like rates**. This name is justified by Theorems 4.1 and 4.2 below. Other good properties of the fraction-like rates are contained in Theorems 4.3–4.6.

Theorem 4.1. *The fraction-like rates comply with the decomposition condition* H *of* §1.3.

Proof. Let us recall that condition H is about unanimously preferred sets, i.e. subsets X of options with the property that each member of X is unanimously preferred to any alternative from outside X. In the following, we will compare such a set X with the top dominant irreducible component of the projected Llull matrix (v_{xy}^π), whose existence is guaranteed by Proposition 3.3. The latter set will be denoted as $\widehat{X}$.

Part (H1). *If X is an unanimously preferred set, then the fraction-like rates vanish outside X.* One easily sees (as in $[2\!:\!\mathrm{Lem.\ 7.1}]$) that the hypothesis that $v_{xy} = 1$ for all $x \in X$ and $y \notin X$ results in the following facts for all such pairs: $v_{xy}^\pi = v_{xy}^* = v_{xy} = 1$; $v_{yx}^\pi = v_{yx}^* = v_{yx} = 0$. This implies that $\widehat{X} \subseteq X$, which leads to the claimed conclusion since Theorem 2.2 ensures that $\varphi_y = 0$ for any $y \notin \widehat{X}$.

Part (H2). *Here we assume that the individual votes are complete, or alternatively, that each of them is a ranking (possibly truncated or with ties). If X is a minimal unanimously preferred set, then the fraction-like rates of X are all of them positive and they coincide with those that one obtains when the individual votes are restricted to X.*

Let us begin by noticing that the CLC structure of the projected Llull matrix ensures that $t_{xy}^\pi = 1$ for all $x, y \in X$. In fact, the inequalities (43) allow to derive it from the known fact —obtained in part (H1)— that $t_{xy}^\pi = 1$ for all $x \in X$ and $y \notin X$.

Now we claim that under the present hypotheses, i.e. either completeness or ranking character of the individual votes, one has $\widehat{X} = X$. In fact, a strict inclusion $\widehat{X} \subset X$ would mean that $v_{x\hat{x}}^\pi = 0$ for any $x \in X \setminus \widehat{X}$ and $\hat{x} \in \widehat{X}$. By the remark of the preceding paragraph, this implies that $v_{\hat{x}x}^\pi = 1$ for all such pairs. Since we also have $v_{xy}^\pi = 1$ for $x \in X$ and $y \notin X$, we can conclude that $v_{\hat{x}\hat{y}}^\pi = 1$ for all $\hat{x} \in \widehat{X}$ and $\hat{y} \notin \widehat{X}$. Now, according to $[1\!:\!\mathrm{Lem.\ 9.1}]$ (for the complete case) and $[2\!:\!\mathrm{Lem.\ 7.1}]$ (for the case of rankings, which are certainly transitive), this implies that $v_{\hat{x}\hat{y}} = 1$ for all such pairs. This contradicts the supposed minimality of X.

So, X itself is the top dominant irreducible component of the matrix (v_{xy}^π). By making use of Theorem 2.2, it follows that $\varphi_x > 0$ for all $x \in X$ and that they are the strengths determined by the restriction of the projected

Llull matrix (v_{xy}^π) to the set X. In order to arrive at the statement of (H2), we must show that this restriction of the projected Llull matrix coincides with the projection of the same restriction applied to the original Llull matrix (v_{xy}), i.e. $v_{x\bar{x}}^\pi = \widetilde{v}_{x\bar{x}}^\pi$ for any $x, \bar{x} \in X$, where we are using a tilde to denote the objects associated with the matrix obtained by first restricting and then projecting. In order to establish this equality, it suffices to obtain analogous equalities for the corresponding margins and turnouts. Besides, by taking into account the way that the CLC projection is defined, it suffices to obtain these equalities for $\bar{x} = x'$, namely the option that immediately follows x in an admissible order ξ (X is easily seen to be a segment of ξ). For the margins, this equality is obtained in $[1{:}\,\text{Lem.}\ 9.2]$, whose proof is valid without any need for completeness. For the turnouts, this equality is immediately true in the complete case. In the case of ranking votes, it suffices to observe that the X restriction of the original Llull matrix is complete. This is true because of the following implications: (i) $v_{xy} = 1$ for some $y \in A$ implies that x is explicitly mentioned in all of the ranking votes; and (ii) x being explicitly mentioned in all of the ranking votes implies that $t_{xy} = 1$ for any $y \in A$.

Part (H3). *In the complete case the converse implication to that of* (H2) *holds too: If the fraction-like rates of X are all of them positive and they coincide with those that one obtains when the individual votes are restricted to X, then X is a minimal unanimously preferred set.*

Let us begin by noticing that the present hypothesis implies that $\sum_{x \in X} \varphi_x = 1$, from which it follows that $\varphi_y = 0$ for all $y \in Y = A \setminus X$. Now, this equality together with the inequality $\varphi_x > 0$ for all $x \in X$ implies that X is the top dominant irreducible component of the matrix (v_{xy}^π). In fact, otherwise Theorem 2.2 would imply the existence of some $x \in X$ with $\varphi_x = 0$ or some $y \in Y$ with $\varphi_y > 0$. In particular, we have $v_{yx}^\pi = 0$ for all $x \in X$ and $y \in Y$. Because of the completeness assumption, this implies that $v_{xy}^\pi = 1$ and —by $[2{:}\,\text{Lem.}\ 7.1]$— $v_{xy} = 1$ for all those pairs.

Finally, let us see that X is minimal for this property: If we had $\widehat{X} \subset X$ satisfying $v_{\hat{x}\hat{y}} = 1$ for all $\hat{x} \in \widehat{X}$ and $\hat{y} \in \widehat{Y} = A \setminus \widehat{X}$, then $[1{:}\,\text{Lem.}\ 9.1]$ would give $v_{\hat{x}\hat{y}}^\pi = 1$ and therefore $v_{\hat{y}\hat{x}}^\pi = 0$ for all such pairs, so X could not be the top dominant irreducible component of the matrix (v_{xy}^π). □

Theorem 4.2. *The fraction-like rates comply with the single-choice voting condition* I*: When each ballot reduces to choosing a single option, the fraction-like rates reduce to the obtained vote fractions.*

Proof. As it was mentioned in §1.3, in the case of single-choice voting one has $v_{xy} = f_x$, and therefore $t_{xy} = f_x + f_y$, for every $y \neq x$. Such matrices have

CLC structure, so they are invariant by the CLC projection: $v_{xy}^\pi = v_{xy} = f_x$. By plugging these values in (15–16), one easily sees that these equations are satisfied by taking $\varphi_x = f_x$. $\qquad\square$

Theorem 4.3. *The fraction-like rates depend continuously on the original Llull matrix.*

Proof. This is a consequence of the continuity of the mappings P and Z. The former is guaranteed by [2 : Thm. 6.1] and the latter by Theorems 2.2 and 2.1. $\qquad\square$

Theorem 4.4. *The fraction-like rates comply with the following majority principle: If A is partitioned in two sets X and Y with the property that $v_{xy} > 1/2$ for any $x \in X$ and $y \in Y$, then for any such x and y one has either $\varphi_x > \varphi_y$ or $\varphi_x = \varphi_y = 0$.*

Proof. Let us assume that the original Llull matrix is in the situation considered by the stated majority principle. According to [2 : Thm. 8.1], the mean preference scores of the projected Llull matrix satisfy then the inequality $\rho_x > \rho_y$ for any $x \in X$ and $y \in Y$ (instead of the mean preference scores ρ_x, in [2] we were dealing with the mean ranks $\bar{r}_x$, which are related to ρ_x by the linear decreasing transformation (6)). So, it suffices to combine that result with Theorem 3.5 of the preceding section. $\qquad\square$

Theorem 4.5. *The fraction-like rates comply with the following clone consistency condition: Assume that $C \subset A$ is an autonomous set for each of the individual votes. Assume also that either $C \subseteq X$ or $C \supseteq A \setminus X$, where $X = \{ x \in A \mid \varphi_x > 0 \}$. Under these hypotheses one has the following facts: (a) C is autonomous for the ranking determined by the fraction-like rates; and (b) contracting C to a single option in all of the individual votes has no other effect in that ranking than getting the same contraction.*

A set $C \subseteq A$ being autonomous for a binary relation η means that every element from outside C relates in the same way to all elements of C; more precisely, for any $x \notin C$, having $ax \in \eta$ for some $a \in C$ implies $bx \in \eta$ for any $b \in C$, and similarly, having $xa \in \eta$ for some $a \in C$ implies $xb \in \eta$ for any $b \in C$. In such a situation, it makes sense to keep considering the binary relation η after contracting the set C, i. e. replacing it by a single element.

Proof. It suffices to combine [2 : Thm. 8.2] with Theorem 3.5 of the preceding section. $\qquad\square$

Theorem 4.6. *Assume that the scores v_{xy} are modified into new values $\widetilde{v}_{xy}$ such that*

$$\widetilde{v}_{ay} \geq v_{ay}, \quad \widetilde{v}_{xa} \leq v_{xa}, \quad \widetilde{v}_{xy} = v_{xy}, \qquad \forall x, y \neq a. \qquad (63)$$

In these circumstances the fraction-like rates behave in the following way:
$\varphi_a > \varphi_y \implies \widetilde{\varphi}_a \geq \widetilde{\varphi}_y.$

Proof. According to [2 : Thm. 8.3], the mean preference scores behave in the following way: $\rho_a > \rho_y \implies \widetilde{\rho}_a \geq \widetilde{\rho}_y$. So, it suffices to combine that result with Theorem 3.5 of the present article. $\square$

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